RELATIONSHIPS BETWEEN REDUCIBILITIES(¹)

BY

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1. **Introduction.** We shall consider the following recursion theoretic reducibilities: one-one (1), many-one (m), truth-table (tt), and Turing (T). These were introduced by Post [13] and are treated also by Rogers in [15]. If R is one of these reducibilities, we write $A \leq_R B$ if A is R-reducible to B and $A \equiv_R B$ if $A \leq_R B$ and $B \leq_R A$. Also we write $B \not\leq_R A$ if $B \leq_R A$ is false, and $A <_R B$ if $A \leq_R B$ and $B \not\leq_R A$. For any set B, the collection of all A such that $A \equiv_R B$ is an *R-degree*. An R-degree is *r.e.* (*recursive*) if it contains at least one r.e. (recursive) set.

Myhill [11, Theorems 5 and 16] proved that the m-degree of the creative sets consists of a single 1-degree. Young [18, Corollary 1.5] showed that every non-recursive m-degree either consists of a single 1-degree or contains a collection of 1-degrees having the order type of the rationals. He asked whether there exist r.e. nonrecursive m-degrees consisting of a single 1-degree other than the m-degree of the creative sets. We show here that the answer is affirmative and indeed that each r.e. T-degree contains an r.e. m-degree consisting of a single 1-degree. (This has also been proved by Soare [16, Corollary 4.6].) We also show that each r.e. nonrecursive T-degree contains infinitely many r.e. m-degrees and each non-recursive tt-degree contains infinitely many m-degrees. Finally we show that each T-degree either consists of a single tt-degree or contains infinitely many tt-degrees.

We assume that the collection of all finite sets has been effectively coded onto the set of natural numbers, and we write D_x for the finite set with index x. We freely *identify* x and D_x, and if $\mathscr{A}$ is a collection of finite sets, we often view $\mathscr{A}$ as $\{x : D_x \in \mathscr{A}\}$. If A is a set, we write $\mathscr{A}^\omega$ for $\{D_x : D_x \subset A\}$.

If g is a recursive function such that for all x, $x \in C \Leftrightarrow g(x) \in B$, we say $C \leq_m B$ *via g*. We say that the array $D_{f(x)}$ *witnesses* that B is not hyperimmune if f is a recursive function, $D_{f(x)} \cap D_{f(u)} = \varnothing$ for $x \neq u$, and $D_{f(x)} \cap B \neq \varnothing$ for all x. We write $\overline{A}$ for the complement of the set A and $|D_x|$ for the cardinality of the finite set D_x.

2. **R.e. m-degrees and 1-degrees within r.e. T-degrees.** The following technical lemma will be useful in the proof of both theorems of this section.

LEMMA 1. *Suppose A is simple but $D_{f(x)}$ witnesses that $\overline{A}$ is not hyperimmune.*

Then there is no recursive class $\mathscr{C}$ of nonempty finite sets such that each $D_{f(x)}$ belongs to $\mathscr{C}$ and for all x, y:

(i) $D_x \in \mathscr{C}$, $y \in A \Rightarrow (D_x - \{y\}) \in \mathscr{C}$.

Proof. Assume there were such a class $\mathscr{C}$. Let $\mathscr{M}$ be the collection of all finite sets which belong to $\mathscr{C}$ but have no proper subset which belongs to $\mathscr{C}$. Let $G = \bigsqcup \mathscr{M}$, the union of all the sets in $\mathscr{M}$. It is claimed that G is an infinite r.e. subset of $\bar{A}$. If this claim is proved, the simplicity of A will be contradicted.

Since $\mathscr{C}$ is recursive, $\mathscr{M}$ is recursive, and thus G is r.e. Now any member of $\mathscr{C}$ which intersects A has a proper subset in $\mathscr{C}$ by condition (i). Hence every set in $\mathscr{M}$ is contained in $\bar{A}$, so $G \subset \bar{A}$. Finally note that each set $D_x \in \mathscr{C}$ has a subset $D_z \in \mathscr{M} : D_z$ is any subset of D_x of minimal cardinality among the subsets of D_x which belong to $\mathscr{C}$. Thus each $D_{f(x)}$ has a subset in $\mathscr{M}$ and therefore G intersects each $D_{f(x)}$, since all members of $\mathscr{C}$ are nonempty. Hence G is infinite.

THEOREM 2. *If A is simple but not hypersimple and $\mathscr{A}^{\omega} \leqq_m B$, then $\mathscr{A}^{\omega} \leqq_1 B$.*

Proof. Assume $\mathscr{A}^{\omega} \leqq_m B$ via the recursive function g, i.e. for all x, $D_x \subset A \Leftrightarrow g(x) \in B$. Consider a propositional language $\mathscr{L}$ with all the standard connectives in which the statement letters are "$y \in A$" and "$y \in B$" for each number y. The assumptions that have been made can be represented in $\mathscr{L}$ by the following infinite but r.e. set of axioms T:

(1_x) $\qquad\qquad\qquad D_x \cap \bar{A} \neq \varnothing \leftrightarrow g(x) \notin B \qquad$ all x.

(2_y) $\qquad\qquad\qquad y \in A \qquad\qquad\qquad\qquad\qquad$ all $y \in A$.

(3_x) $\qquad\qquad\qquad D_{f(x)} \cap \bar{A} \neq \varnothing \qquad\qquad\qquad$ all x.

(Any statement above which is not a formula of $\mathscr{L}$ should be interpreted in the obvious way. For instance, if $D_{f(0)} = \{2, 5\}$, then "$D_{f(0)} \cap \bar{A} \neq \varnothing$" is rendered in $\mathscr{L}$ by "$\neg 2 \in A \vee \neg 5 \in A$". Note also that we use $\leftrightarrow$ for equivalence in $\mathscr{L}$ while $\Leftrightarrow$ is used metalinguistically.)

If ϕ is (an abbreviation for) a formula of $\mathscr{L}$, $\vdash_T \phi$ means that ϕ can be proved from the hypotheses T within the propositional calculus.

For each number k, let

$$E_k = \{y : \vdash_T D_k \subset A \leftrightarrow y \in B\}.$$

We assume temporarily that E_k is infinite for every k and show how the theorem then follows. Notice that the sets E_k are r.e. uniformly in k. To define the recursive function h by induction, let $h(k)$ be the first number y which occurs in a uniform enumeration of E_k such that $y \neq h(k')$ for all $k' < k$. h is total because E_k is always infinite, and so $\mathscr{A}^{\omega} \leqq_1 B$ via h by the soundness of propositional logic.

It remains to show that each E_k is infinite. We fix k and assume that E_k is finite. Consider first the case where $\vdash_T D_k \cap \bar{A} \neq \varnothing$. Then $E_k = \{y : \vdash_T y \notin B\}$. Thus, by axioms (1_x), $\vdash_T D_x \cap \bar{A} \neq \varnothing$ iff $g(x) \in E_k$, for any x. Hence if $\mathscr{C} = \{D_x : \vdash_T D_x \cap \bar{A} \neq \varnothing\}$, $\mathscr{C}$ is recursive. Also $\mathscr{C}$ contains only nonempty sets by the soundness of propositional logic, and all the sets $D_{f(x)}$ belong to $\mathscr{C}$ by axioms (3_x). Finally,

condition (i) of Lemma 1 is satisfied for $\mathscr{C}$ because of axioms (2_y). Hence the assumption that E_k was finite has led to a contradiction in this case.

Now assume that $\nvdash_T D_k \cap \bar{A} \neq \varnothing$. (We write "$\nvdash_T \phi$" for "not $\vdash_T \phi$.") Let

$$\mathscr{C}_k = \{D_x : \nvdash_T D_x \cup D_k \subset A \leftrightarrow D_k \subset A\}.$$

Clearly $\mathscr{C}_k$ consists only of nonempty sets. Also $\mathscr{C}_k$ contains all the sets $D_{f(x)}$ because if $D_{f(x)} \notin \mathscr{C}_k$ for some particular x, then $\vdash_T D_{f(x)} \cup D_k \subset A \leftrightarrow D_k \subset A$ and so $\vdash_T D_k \cap \bar{A} \neq \varnothing$ by axiom (3_x). To see that $\mathscr{C}_k$ satisfies condition (i) of Lemma 1 we assume that $y \in A$ and $(D_x - \{y\}) \notin \mathscr{C}_k$ and prove that $D_x \notin \mathscr{C}_k$. Since $D_x - \{y\} \notin \mathscr{C}_k$,

$$\vdash_T (D_x - \{y\}) \cup D_k \subset A \leftrightarrow D_k \subset A.$$

Since $y \in A$ we see from axiom (2_y) that

$$\vdash_T D_x \cup D_k \subset A \leftrightarrow D_k \subset A,$$

i.e. $D_x \notin \mathscr{C}_k$.

To verify that $\mathscr{C}_k$ is recursive we apply axioms (2_x) to the definition of $\mathscr{C}_k$ and deduce that for all x,

$$D_x \in \mathscr{C}_k \Leftrightarrow \nvdash_T g(D_x \cup D_k) \in B \leftrightarrow D_k \subset A.$$

Thus $D_x \in \mathscr{C}_k$ iff $g(D_x \cup D_k) \notin E_k$. Since E_k is finite, it follows that $\mathscr{C}_k$ is recursive and we have a contradiction to Lemma 1.

The following corollary has been proved independently by Soare [16, Corollary 4.6].

COROLLARY 3. *Each r.e. T-degree contains an r.e. m-degree which consists of a single 1-degree.*

Proof. In the recursive T-degree, the m-degree of $\varnothing$ consists of a single 1-degree. By a theorem of Yates, [17, Theorem 2], each nonrecursive r.e. T-degree a contains a simple nonhypersimple set A. Then $\mathscr{A}^\omega$ is also an r.e. set of degree a, and we claim that the m-degree of $\mathscr{A}^\omega$ consists of a single 1-degree. Assume $\mathscr{A}^\omega \equiv_m B$. By Theorem 2, $\mathscr{A}^\omega \leq_1 B$. Because A is an infinite r.e. set it is easy to see from Lemma 0.5 of [18] that $\mathscr{A}^\omega$ is a cylinder. It then follows from [15, Theorem VIII(c)] that $B \leq_1 \mathscr{A}^\omega$. Thus $B \equiv_1 \mathscr{A}^\omega$ and we have the desired conclusion.

We remark that in Theorem 2, $\mathscr{A}^\omega$ may be replaced by $\mathscr{A}^\omega \times B$, where B is any nonempty set. ($C \times B$, for sets C, B, is the image of the cartesian product of C and B under a recursive pairing function.) The proof of Theorem 2 remains essentially the same. The reasoning of the proof of Corollary 3 then shows that each T-degree which has a nonrecursive r.e. predecessor has an m-degree consisting of a single 1-degree. We do not know whether *all* T-degrees have an m-degree consisting of a single 1-degree.

If A is a set, we let $\mathscr{A}^n = \{D_x : D_x \subset A \ \& \ |D_x| \leq n\}$. Note that $\mathscr{A}^n \leq_m \mathscr{A}^{n+1}$ for all n. The theorems we shall state for $\mathscr{A}^n$ we could equally well state for A^n

($=A \times \cdots \times A$ (n factors)), since $A^n \equiv_m \mathscr{A}^n$ for all $n > 0$. Stating the theorems in terms of A^n is more conventional, but $\mathscr{A}^n$ is more convenient technically in the present context.

THEOREM 4. *If A is simple but not hypersimple, then for all n, $\mathscr{A}^n <_m \mathscr{A}^{n+1}$.*

Proof. Assume, for *reductio ad absurdum*, that $\mathscr{A}^{n+1} \leq_m \mathscr{A}^n$. Call two sets E, E' *A-equivalent* if $E \subset A \Leftrightarrow E' \subset A$. The assumption that $\mathscr{A}^{n+1} \leq_m \mathscr{A}^n$ implies that given any set of cardinality $n+1$ one can effectively find an A-equivalent set of cardinality at most n. Thus given any finite set E of cardinality $k > n$, one may effectively find an A-equivalent set F of cardinality at most $k-1$ by replacing some $n+1$ element subset of E by an A-equivalent set of at most n elements. If this procedure is iterated $k-n$ times, one finds effectively a set of cardinality at most n which is A-equivalent to E. Hence $\mathscr{A}^\omega \leq_m \mathscr{A}^n$.

Assume $\mathscr{A}^\omega \leq_m \mathscr{A}^n$ via g, where $|D_{g(x)}| \leq n$ for all x. Assume that $D_{f(x)}$ witnesses that $\overline{A}$ is not hyperimmune. Let $\mathscr{L}$ be the propositional language which has statement letters "$y \in A$" for each number y. Let T be the following r.e. set of axioms in $\mathscr{L}$:

(1_x) $\qquad\qquad$ $D_x \cap \overline{A} \neq \varnothing \leftrightarrow D_{g(x)} \cap \overline{A} \neq \varnothing$ $\qquad$ all x.

(2_y) $\qquad\qquad$ $y \in A$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ all $y \in A$.

(3_x) $\qquad\qquad$ $D_{f(x)} \cap \overline{A} \neq \varnothing$ $\qquad\qquad\qquad\qquad$ all x.

We use "$\vdash_T \phi$" as in Theorem 2. For each k, let $\mathscr{A}_k = \{D_x : |D_x| \leq k \ \& \ \vdash_T D_x \cap \overline{A} \neq \varnothing\}$. We prove that $\mathscr{A}_k$ is recursive by induction on k. $\mathscr{A}_0$ is the empty class. Now assume that $\mathscr{A}_k$ is recursive and consider $\mathscr{A}_{k+1}$. We claim that $\mathscr{A}_{k+1}$ contains only finitely many sets which have no subset in $\mathscr{A}_k$. The proof of this is similar to the proof of Lemma 1. Let $\mathscr{M}$ be the collection of all sets in $\mathscr{A}_{k+1}$ which have no subset in $\mathscr{A}_k$ and let $G = \bigcup \mathscr{M}$. $\mathscr{M}$ is r.e. because $\mathscr{A}_{k+1}$ is r.e. and $\mathscr{A}_k$ is recursive. Hence G is r.e. It is easy to show from axioms 2_x that $G \subset \overline{A}$. Hence G is finite, so $\mathscr{M}$ is finite. Thus $\mathscr{A}_{k+1}$, with finitely many exceptions, consists just of those sets which have at most $k+1$ elements and contain some set in $\mathscr{A}_k$. Hence, by another application of the induction assumption, $\mathscr{A}_{k+1}$ is recursive.

Let $\mathscr{C} = \{D_x : \vdash_T D_x \cap \overline{A} \neq \varnothing\}$. By axiom 1_x, $D_x \in \mathscr{C} \Leftrightarrow D_{g(x)} \in \mathscr{A}_n$. Since $\mathscr{A}_n$ is recursive, it follows that $\mathscr{C}$ is recursive. One many verify exactly as for $\mathscr{C}$ in Theorem 2 that $\mathscr{C}$ satisfies the other hypotheses of Lemma 1. But Lemma 1 states that no such $\mathscr{C}$ exists, and so we have a contradiction.

Fischer [4, Theorem 2] first proved Theorem 4 (for A^n) in the special case where A was Post's *tt*-complete simple set [13, p. 304]. However, our proof is entirely different from his.

COROLLARY 5. *Each nonrecursive r.e. T-degree contains an infinite chain of r.e. m-degrees.*

Proof. Let a be a given nonrecursive r.e. T-degree, and let A be a simple non-hypersimple set of T-degree a [17, Theorem 2]. Then the r.e. sets $\mathscr{A}^1, \mathscr{A}^2, \ldots$ also have T-degree a and $\mathscr{A}^1 <_m \mathscr{A}^2 <_m \cdots$ by the theorem.

We remark that the assumption of nonhypersimplicity cannot be dropped from either theorem of this section. In particular, it is easy to see from [6, Theorems 3.2 and 4.2 (i)] and well-known results that the conclusions of these theorems are false for any set A obtained by Dekker's hypersimple set construction [2, Theorem 1].

It is natural to enquire whether an analogue of Corollary 5 holds for *antichains*, i.e. collections of mutually incomparable m-degrees. Along these lines, Lerman [8, Theorem 5.1] proved that each r.e. T-degree a with $a' = 0''$ has an infinite antichain of m-degrees of maximal sets. The m-degrees of maximal sets have the additional interesting property of being minimal among the nonrecursive m-degrees [7, p. 4] and [15, p. 239]. Lachlan [7, Theorem 4] has shown the existence of a minimal r.e. m-degree in each nonzero r.e. T-degree, but the existence of an infinite antichain of r.e. m-degrees in each such T-degree has not yet been established.

3. m-degrees within tt-degrees.

THEOREM 6. *If B is immune but not hyperimmune and retraced by a general recursive function and $A = \bar{B}$, then for all n, $\mathscr{A}^n <_m \mathscr{A}^{n+1}$.*

Proof. Assume that B is retraced by the general recursive function h. This means [3, p. 358] that if b_k is the kth smallest member of B, then $h(b_0) = b_0$ and $h(b_{k+1}) = b_k$ for all $k \geq 0$. We may assume that $h(x) \leq x$ for all x, i.e that h is *downward*. The proof to follow may best be visualized in terms of the "retracing tree" of h. (See [3, p. 359].) Following Appel and McLaughlin [1, p. 83] we set $\hat{h}(y) = \{h^k(y) : k \geq 0\}$. (Here $h^0(y) = y$ and $h^{k+1}(y) = h(h^k(y))$.) Because h is general recursive and downward, the predicate $z \in \hat{h}(y)$ is recursive.

We say that y and z are *h-comparable* if $y \in \hat{h}(z)$ or $z \in \hat{h}(y)$. If y and z both belong to B, y and z are h-comparable. We write $y \,|\, D_x$ if y is not h-comparable with any member of D_x.

Let $A = \bar{B}$ and assume that $\mathscr{A}^{n+1} \leq_m \mathscr{A}^n$ via g. We may assume that $|D_{g(x)}| \leq n$ whenever $|D_x| \leq n+1$, so we have for such x,

$$D_x \subset \bar{B} \Leftrightarrow D_{g(x)} \subset \bar{B}.$$

We now wish to define a certain r.e. set $C \subset \bar{B}$. Let $C = C_1 \cup C_2$, where

$$C_1 = \{y : (\exists x)[|D_x| \leq n+1 \ \& \ y \in D_x \ \& \ y \,|\, D_{g(x)}]\},$$
$$C_2 = \{y : (\exists x)[|D_x| \leq n+1 \ \& \ y \,|\, D_x \ \& \ \hat{h}(y) \cap D_{g(x)} \neq \varnothing]\}.$$

C_1 and C_2 are r.e. so C is r.e. To see that $C_1 \subset \bar{B}$, suppose that some number y were in $C_1 \cap B$. Let x be such that $|D_x| \leq n+1$ and $y \in D_x$ and $y \,|\, D_{g(x)}$. Since $y \in B$ and $y \,|\, D_{g(x)}$, $D_{g(x)} \subset \bar{B}$. Thus also $D_x \subset \bar{B}$, but this is impossible since $y \in D_x \cap B$.

To see that $C_2 \subset \bar{B}$, suppose that some number y were in $C_2 \cap B$. Let x be such that $|D_x| \leqq n+1$ and $y|D_x$ and $\hat{h}(y) \cap D_{g(x)} \neq \varnothing$. Since $y \in B$ and $y|D_x$, $D_x \subset \bar{B}$. Hence $D_{g(x)} \subset \bar{B}$. Since $y \in B$, $\hat{h}(y) \subset B$, so $\hat{h}(y) \cap D_{g(x)} = \varnothing$. This is a contradiction. Thus $C_2 \subset \bar{B}$, so $C \subset \bar{B}$.

For any y, let the *level* of y be the least number k such that $h^k(y) = h^{k+1}(y)$. (k exists because h is downward.) A set D is *level* if all members of D have the same level. We claim that C is so large that $\bar{C}$ has no level subset of cardinality exceeding $2n$. (Recall that n was chosen so that $\mathscr{A}^{n+1} \leqq_m \mathscr{A}^n$.) To facilitate the proof of this claim, a partial ordering $<_l$ of the natural numbers will be defined such that any two numbers y, z are $<_l$-comparable iff y and z have the same level. If y and z have level 0, we define $y <_l z$ iff $y < z$. If y and z have level $k+1$, then $h(y)$ and $h(z)$ have level k, and we assume inductively that we have defined $<_l$ for $h(y)$ and $h(z)$. We now define

$$y <_l z \Leftrightarrow h(y) <_l h(z) \quad \text{or} \quad (h(y) = h(z) \ \& \ y < z).$$

(Thus if the retracing tree of h is drawn appropriately, $y <_l z$ means that y and z have the same level and y lies to the left of z.)

Clearly $<_l$ is a partial ordering. We write $y \leqq_l z$ if $y <_l z$ or $y = z$. If $y <_l z$, then for any k, $h^k(y) \leqq_l h^k(z)$.

Now suppose, contrary to our claim, that $\bar{C}$ has a level subset $\{y_1, y_2, \ldots, y_{2n+1}\}$ of cardinality $2n+1$. Assume $y_1 <_l y_2 <_l \cdots <_l y_{2n+1}$, and let

$$D_x = \{y_1, y_3, y_5, \ldots, y_{2n+1}\}.$$

Since no member of D_x is in C_1, every member of D_x is h-comparable with some member of $D_{g(x)}$. But since $|D_x| \leqq n+1$, $|D_{g(x)}| \leqq n$. Thus some member z of $D_{g(x)}$ is h-comparable with two distinct members, say y_{2i+1} and y_{2j+1}, of D_x. Hence, y_{2i+1} and y_{2j+1} have the same level, there is a k such that $h^k(y_{2i+1}) = h^k(y_{2j+1}) = z$.

Assume $i < j$. It will be shown that $y_{2i+2} \in C_2$. This will give the desired contradiction and prove the claim. Note that $y_{2i+1} <_l y_{2i+2} <_l y_{2j+1}$. Hence $h^k(y_{2i+1}) \leqq_l h^k(y_{2i+2}) \leqq_l h^k(y_{2j+1})$, so $z \leqq_l h^k(y_{2i+2}) \leqq_l z$. It follows that $h^k(y_{2i+2}) = z$.

Now consider the set $D_x = \{y_1, y_3, \ldots, y_{2k+1}\}$. Since $y_{2i+2} \notin D_x$ and $D_x \cup \{y_{2i+2}\}$ is level, $y_{2i+2}|D_x$. Also $z \in \hat{h}(y_{2i+2}) \cap D_{g(x)}$, so $\hat{h}(y_{2i+2}) \cap D_{g(x)} \neq \varnothing$. Thus x witnesses that $y_{2i+2} \in C_2$, which was to be shown.

Let $b_0, b_1, \ldots$ be the members of B in increasing order. Since B is not hyperimmune, it follows from [14, Theorem 21] that there is a recursive function f such that $f(k) > b_k$ for all k. We now define a recursive sequence of sets $D_{p(k)}$ all intersecting B. To find $D_{p(k)}$, list C until there are at most $2n$ numbers of level k which are less than $f(k)$ and have not yet appeared in C. By the previous argument, this state of affairs must eventually be reached for every k. Then let $D_{p(k)}$ be those numbers $< f(k)$ of level k which have not yet appeared in C when this state of affairs is reached. Thus $|D_{p(k)}| \leqq 2n$ for all k. Also $D_{p(k)} \cap B \neq \varnothing$ because $b_k \in D_{p(k)} \cap B$. Finally the $D_{p(k)}$ are pairwise disjoint because the levels are. However,

Theorem 2 of [1] states that no immune retraceable set is witnessed nonhyper-immune by an array (such as $D_{p(k)}$) of sets of bounded cardinality. This contradiction completes the proof.

We remark as in §2 that the conclusion of Theorem 6 fails even for $n=1$ when B is permitted to be hyperimmune and indeed whenever B is the complement of some hypersimple set A obtained by Dekker's hypersimple set construction. On the other hand, if $\mathscr{A}^2 \leq_m \mathscr{A}^1$ where $B = \overline{A}$ is retraceable, then A must be r.e. [5, Corollary 4.29]. It now follows from [3, Theorem 4] that $\mathscr{A}^1 <_m \mathscr{A}^2$ when $\overline{A}$ is immune, nonhyperimmune, and retraceable. We conjecture that under these assumptions $\mathscr{A}^n <_m \mathscr{A}^{n+1}$ for all n, so that the existence of a *general recursive* retracing function is not necessary in Theorem 6.

COROLLARY 7. *Each nonrecursive tt-degree contains an infinite chain of m-degrees.*

Proof. We need to know that each nonrecursive tt-degree contains an immune nonhyperimmune set B retraced by a general recursive function. This result is implicit in [3]. Suppose we are given a set E. Let $\{a_n\}$ be the sequence of values of the characteristic function of E and let $\sigma = \Phi\langle a_n \rangle$ be the set constructed from $\{a_n\}$ on p. 363 of [3]. Then it is pointed out in [3] that if E is nonrecursive, then σ is immune but not hyperimmune. Further, it is easy to check that $\sigma \equiv_{tt} E$ and σ is retraced by a general recursive function. Hence, if we let $B = \sigma$ in the theorem, we see that $\mathscr{A}^n <_m \mathscr{A}^{n+1}$, where $A = \overline{B}$. Also we have $E \equiv_{tt} \sigma \equiv_{tt} A \equiv_{tt} \mathscr{A}^n$ for all n.

The author does not know whether Corollary 7 holds for infinite antichains, but the existence of a *pair* of incomparable m-degrees within each tt-degree is proved in [6, Corollary 4.3(iv)].

4. tt-degrees within T-degrees. Martin has communicated to the author a sufficient condition for a T-degree to consist of a single tt-degree. In the following theorem we show that his condition is also necessary and deduce that each T-degree either consists of a single tt-degree or contains an infinite chain of tt-degrees.

THEOREM 8. *If a is any T-degree, the following statements are equivalent*:

(i) *Every function f of degree $\leq a$ is (everywhere) majorized by some recursive function.*

(ii) *a consists of a single tt-degree.*

(iii) *a has a finite number of tt-degrees.*

(iv) *There is a maximum tt-degree among the tt-degrees contained in a.*

Proof. (We remark that by Corollary 1.1 of [10], (i) is equivalent to the assertion that no hyperimmune set has degree a. However, we shall not need this fact.)

In this proof we write $\{e\}^C$ for the eth function partial recursive in the set C. We write $\{e\}_s^C(x) = y$ if at most s machine steps are necessary to compute $\{e\}^C(x) = y$. $\{e\}$ and $\{e\}_s$ are $\{e\}^\varnothing$, $\{e\}_s^\varnothing$ respectively. We also identify sets and their characteristic functions.

The implication (i) $\Rightarrow$ (ii) is due to Martin, but we give it here because it is unpublished. Assume (i). Let A be a set of degree $\boldsymbol{a}$ and assume $B \leq_T A$. We wish to show $B \leq_{tt} A$. Assume $B = \{e\}^A$. Define $f(x) = \mu s \ [\{e\}_s^A(x)$ is defined]. Then f is recursive in A, so by (i) there is a recursive function g such that for all x, $g(x) > f(x)$. Define a new recursive reduction procedure as follows:

$$\{e'\}^C(x) = \{e\}_{g(x)}^C(x) \quad \text{if } \{e\}_{(gx)}^C(x) \text{ defined,}$$
$$= 0 \qquad\qquad \text{otherwise.}$$

Then $B = \{e'\}^A$ also and $\{e'\}^C$ is a total function for every set C. It follows by a theorem of Nerode [12, p. 250] or [15, p. 143] that $B \leq_{tt} A$. Thus if $B \equiv_T A$, we may also apply this argument with A and B interchanged and deduce that $B \equiv_{tt} A$. Hence (ii) holds.

The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are trivial.

We now show that (iv) $\Rightarrow$ (i). Assume that (iv) is true but (i) is false. Thus $\boldsymbol{a}$ has a maximum tt-degree, and within this tt-degree there is a maximum m-degree by [15, p. 113]. Let A be a set belonging to the maximum m-degree in $\boldsymbol{a}$. Also, let f be a function recursive in $\boldsymbol{a}$ which is majorized by no recursive function. We may assume that f is monotone increasing. We use a diagonal construction similar to that in Theorem I of Chapter 9 of [15]. Let

$$B = \{e : \{e\}_{f(e)}(e) \text{ defined \& } \{e\}_{f(e)}(e) \notin A\}.$$

Clearly, $B \leq_T A$. We claim $B \not\leq_m A$. Assume $B \leq_m A$ via $\{e_0\}$. Assume that $e_0, e_1, \ldots$ is a recursive sequence such that $\{e_i\} = \{e_0\}$ and $e_i > i$ for all i. Thus we have for all i, $e_i \in B \Leftrightarrow \{e_i\}(e_i) \in A$. Then we see from the definition of B that $\{e_i\}_{(f(e_i))}(e_i)$ is defined for no i. Thus if we define $g(i) = \mu s \ [\{e_i\}_s(e_i)$ is defined], we have $g(i) \geq f(e_i) > f(i)$. But since g is recursive, this contradicts the selection of f. Hence $B \not\leq_m A$.

Let $C = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$. Clearly $C \equiv_T A$ and $A <_m C$. This contradicts the selection of A.

COROLLARY 9. (i) *If a T-degree consists of a single tt-degree, then so do all lower T-degrees.*

(ii) *Each T-degree either consists of a single tt-degree or contains an infinite chain of tt-degrees.*

Proof. Part (i) of the corollary holds because if (i) of the theorem is true of $\boldsymbol{a}$, it is also true of all T-degrees $\boldsymbol{b} \leq \boldsymbol{a}$. (ii) of the corollary follows immediately from (ii) $\Leftrightarrow$ (iv) in the theorem.

We remark that Miller and Martin [10, Theorem 2.1] have shown the existence of nonrecursive T-degrees which have no hyperimmune sets and hence consist of a single tt-degree. On the other hand, Martin has shown [9, §2, Corollary 1] that the T-degrees of hyperimmune sets have measure 1 so that "almost every" T-degree contains an infinite chain of tt-degrees.

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